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Technical Notes

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Efficient Modal Frequency Response Analysis of Large Structures with Structural Damping

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I. Introduction

RECENT[†] trends of finite element (FE) models in the frequency response analysis (FRA) of large structures such as the automobile, submarine, or aircraft industry reveal heavy use of modal formulation. This is because engineers are creating significantly larger FE models to the limits of their hardware and software.

In the FRA of structures, damping is almost unavoidable [1]. For an undamped system, or a system with proportional viscous damping, the modal frequency response problem becomes uncoupled as a result of the mode orthogonality property and mass normalization, so that it is inexpensive to solve this problem. However, with nonproportional damping, because the modal damping matrices become fully populated [2], it becomes very expensive to solve the modal frequency response problem when nonproportional damping exists. One of the commonly used nonproportional damping materials is structural damping. The purpose of structural damping is to dissipate some energy during each cycle of response to reduce noise and vibration.

The modal frequency response problem with structural damping can be uncoupled with the quadratic eigensolutions. Because solving the quadratic eigenvalue problem (QEP) in its original form is difficult, one needs to linearize it into a generalized eigenvalue problem (GEP), in which the GEP has dimensions that are twice as large as the original QEP [3]. This approach is not appropriate for large scale FE models, which require more than thousands of modes to represent the responses.

Alternatively, the coupled modal frequency response problem, which includes structural damping, has been solved with either direct methods or iterative methods [2]. Direct methods are the most straightforward method, but is expensive for structural systems with many modes due to a factorization cost that is $\mathcal{O}(m^3)$ operations [4], where m is the number of modes used to represent the response and is usually in the thousands for large structures. Iterative methods have a decisive advantage over direct methods in terms of speed and demands of computer memory. However, the convergence rate of iterative methods depends on spectral properties of the coefficient matrix, and the cost increases in proportion to the number of right hand sides when a system has multiple right hand sides [5]. Therefore, there are still disadvantages to solve the damped modal frequency response problem with many modes using the traditional methods.

This paper presents an efficient algorithm, fast frequency response analysis (FFRA) algorithm, to solve the modal frequency response problem for large structures with structural damping. The goal of the FFRA algorithm is $\mathcal{O}(m^2)$ operations at each excitation frequency. The newly developed algorithm FFRA not only greatly improves the performance of the damped modal FRA, but also gives the same accuracy as current existing methods.

II. Damped Modal Frequency Response Problem Formulation

One can represent a system of equations for the direct FRA in the FE dimension as

$$[-\omega^2 M + (1+i\gamma)K + iK_s]\mathbf{X}(\omega) = \mathbf{P}(\omega) \tag{1}$$

where M and $K \in \mathbb{R}^{n\times n}$ are the FE mass matrix and the stiffness matrix, and n represents the number of FE degrees of freedom. The scalar γ is a global structural damping coefficient and $i = \sqrt{-1}$. $K_s \in \mathbb{R}^{n\times n}$ is the FE local structural damping matrix that represents localized deviations of specific elements from the global structural damping level. For excitations $\mathbf{P}(\omega) \in \mathbb{C}^{n\times nf}$, the frequency responses $\mathbf{X}(\omega) \in \mathbb{C}^{n\times nf}$ are calculated at each excitation frequency ω by solving a set of complex linear equation in Eq. (1), where nf represents the number of load cases. Because the large amounts of CPU time, memory, and data transfer are required to solve Eq. (1) for large and complex structural system, modal FRA has been used instead.

The frequency response problem in Eq. (1) is projected onto the subspace spanned by eigenvectors in $\Phi \in \mathbb{R}^{n\times m}$, which is obtained from partial eigensolution of the generalized eigenvalue problem $K\Phi = M\Phi\Lambda$. $\Lambda \in \mathbb{R}^{m\times m}$ is a eigenvalue matrix, and m is the number of modes obtained up to cutoff frequency ($m \ll n$). By making the approximation $\mathbf{X}(\omega) = \Phi\mathbf{Z}(\omega)$ and premultiplying by Φ^T, the modal frequency response problem is obtained in the form

$$[-\omega^2 I + (1+i\gamma)\Lambda + i\bar{K}_s]\mathbf{Z}(\omega) = \mathbf{F}(\omega) \tag{2}$$

as a result of the mode orthogonality property and mass normalization, and $\mathbf{F}(\omega) = \Phi^T\mathbf{P}(\omega)$. Note that the modal structural damping matrix $\bar{K}_s = \Phi^T K_s \Phi \in \mathbb{R}^{m\times m}$ is a fully populated matrix.

III. Fast Frequency Response Analysis for Structural Damping

The goal of the FFRA algorithm for the modal structural damping $\bar{K}_s$ is $\mathcal{O}(m^2)$ operations at each excitation frequency. First, the damped modal frequency response problem in Eq. (2) is rewritten as

$$[-\omega^2 I + C]\mathbf{Z}(\omega) = \mathbf{F}(\omega) \tag{3}$$

in which a complex symmetric matrix C is defined as

$$C = (1+i\gamma)\Lambda + i\bar{K}_s, \qquad C = C^T \in \mathbb{C}^{m\times m} \tag{4}$$

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Note that C is not a Hermitian matrix and is a frequency independent matrix.

Next, the FFRA algorithm solves the following complex symmetric eigenvalue problem for C:

$$C\Phi_C = \Phi_C \Lambda_C \tag{5}$$

where $\Lambda_C \in \mathbb{C}^{m\times m}$ is the diagonal matrix of complex eigenvalues and $\Phi_C \in \mathbb{C}^{m\times m}$ is the corresponding complex eigenvector matrix. Φ_C is normalized to satisfy

$$\Phi_C \Phi_C^T = \Phi_C^T \Phi_C = I \tag{6}$$

and

$$\Phi_C^T C \Phi_C = \Lambda_C \tag{7}$$

To solve the complex symmetric eigenvalue problem efficiently, the algorithm developed by Bar-On [6] is implemented with BLAS 3 operations [7]. The complex SYMMetric eigenvalue problem solver, CSYMM, performs a complex orthogonal transformation using complex symmetric Householder matrices. The complex symmetric Householder matrix H has the form

$$H = I - \tau \boldsymbol{v}\boldsymbol{v}^T \in \mathbb{C} \tag{8}$$

where H is symmetric and orthogonal

$$H^T = H, \qquad H^T H = H H^T = I \tag{9}$$

The τ is defined as

$$\tau = \frac{2}{\boldsymbol{v}^T \boldsymbol{v}} \in \mathbb{C} \tag{10}$$

where τ is different from $\tau = 2/\|\boldsymbol{v}\|_2^2 = 2/(\boldsymbol{v}^H \boldsymbol{v}) \in \mathbb{R}$ in the complex Hermitian matrix case [8]. Note that this paper does not consider a quasi-null case [9], in which a complex vector $\boldsymbol{v} \neq 0$ if and only if $\boldsymbol{v}^T \boldsymbol{v} = 0$, for the scope of journal.

Once the eigensolution of complex symmetric matrix C is obtained, we let

$$\mathbf{Z}(\omega) = \Phi_C \mathbf{W}(\omega) \tag{11}$$

Substituting Eq. (11) into Eq. (3) and premultiplying by Φ_C^T gives

$$\Phi_C^T[-\omega^2 I + C]\Phi_C \mathbf{W}(\omega) = \Phi_C^T \mathbf{F}(\omega) \tag{12}$$

Combining Eqs. (6) and (7) with Eq. (12), Eq. (12) can be written as

$$[-\omega^2 I + \Lambda_C]\mathbf{W}(\omega) = \Phi_C^T \mathbf{F}(\omega) \tag{13}$$

Note that the coefficient matrix becomes a diagonal matrix, $D(\omega) = (-\omega^2 I + \Lambda_C)$, which is frequency dependent. One can easily compute the solution $\mathbf{W}(\omega)$ as follows:

$$\mathbf{W}(\omega) = D(\omega)^{-1} \Phi_C^T \mathbf{F}(\omega) \tag{14}$$

Finally, the modal responses $\mathbf{Z}(\omega)$ can be obtained from the backtransformation in Eq. (11), and then $\mathbf{X}(\omega) = \Phi \mathbf{Z}(\omega)$ is performed for the responses in FE space. We should note that the FFRA algorithm is an efficient reformulation of the modal frequency response problem, not an approximation approach.

The FFRA algorithm requires $\mathcal{O}(m^3)$ operations only one time to solve the complex symmetric eigenvalue problem for C in Eq. (5) before the modal FRA begins. At each frequency, $\mathcal{O}(m^2 \times nf)$ operations are required to form $\Phi_C^T \mathbf{F}(\omega)$. If forces are frequency independent, compute $\Phi_C^T \mathbf{F}(\omega)$ only once before the frequency sweep. Then, $\mathcal{O}(m \times nf)$ operations, which are trivial, are necessary to compute $\mathbf{W}(\omega)$. Finally, the backtransformation requires $\mathcal{O}(m^2 \times nf)$ operations in Eq. (11). Therefore, the FFRA algorithm provides tremendous performance improvement with $\mathcal{O}(m^2)$ operations per load case at each frequency compared with a conventional approach that factors the coefficient matrix at each frequency with $\mathcal{O}(m^3)$ operations per load case at each frequency.

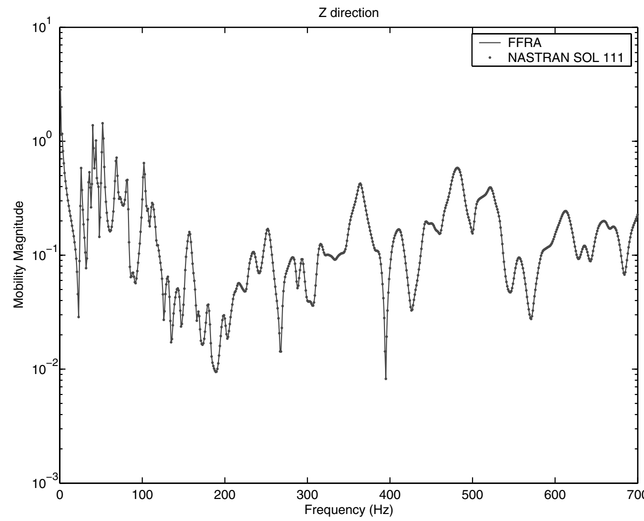


Fig. 1 Comparison of responses from the FFRA and NASTRAN modal solution (SOL 111) for FE model 1.

IV. Numerical Examples

As numerical examples, two industry FE models are used to evaluate the performance and accuracy of the FFRA algorithm compared with a well known commercial FE software NASTRAN [10] which factors the coefficient matrix of a complex linear system for the modal frequency response problem with structural damping.

A. Example 1

A FE model with 114,219 degrees of freedom has 1381 global modes including six rigid body modes obtained from the partial eigensolution. The excitation frequency range is from 1 to 700 Hz. An HP rx5670 with 900 MHz Itanium II processor is used for evaluating the performance of the algorithm. The analysis time of the FFRA algorithm is as follows:

1) Elapsed time of FFRA: 48 s.

2) Elapsed time of NASTRAN modal FRA (SOL 111): 18 min, 37 s.

The FFRA algorithm is 23.3 times faster than NASTRAN modal solution analysis (SOL 111). For the accuracy evaluation, the solution of the FFRA algorithm is compared with that of NASTRAN as shown in Fig. 1, in which both FFRA and NASTRAN modal solution provide the same result.

B. Example 2

The performance of the FFRA algorithm is tested for a trim body car FE model which has 1.58×10^6 degrees of freedom. 5818 global modes are obtained and the frequency range of interest is from 1 to 500 Hz with 1 Hz increment. The number of load cases is three. The FFRA algorithm is almost 35 times faster than NASTRAN modal solution [10] approach:

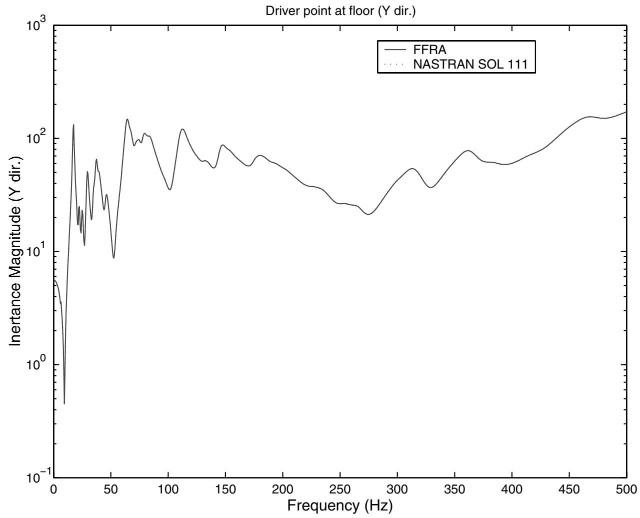
1) Elapsed time of FFRA: 20 min, 11 s.

2) Elapsed time of NASTRAN modal FRA (SOL 111): 11 h, 48 min.

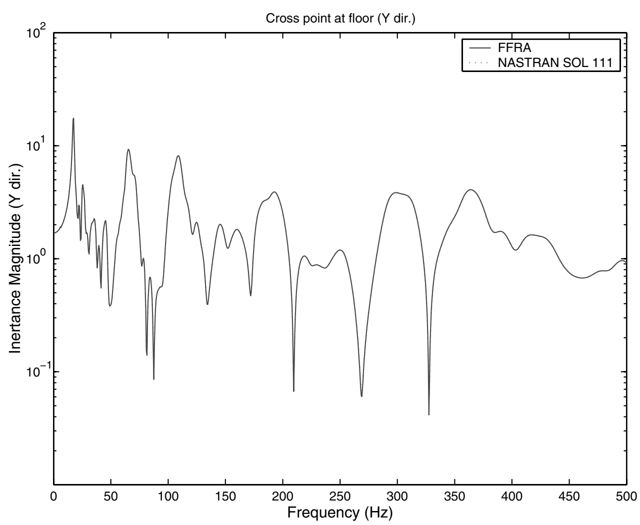
Table 1 represents the timing profile of the main steps in the FFRA algorithm for model 2. Most of the time, 77.8%, is used for solving

Table 1 Timing profile of the algorithm FFRA for FE model 2

Step	Task	Time (mm:ss)	Portion
(1)	CSYMM : $C\Phi_C = \Phi_C \Lambda$	15:35	77.8%
	for $i = 1, N_{\mathrm{freq}}$		
(2.1)	$\mathcal{P} = \Phi_C^T \mathbf{F}$	02:09	10.7%
(2.2)	$\mathbf{W} = D^{-1}\mathcal{P}$	00:03	0.2%
(2.3)	$\mathbf{Z} = \Phi_C \mathbf{W}$	02:11	10.8%
	end		
Total		20:11	



a) Drive point



b) Cross point

Fig. 2 Comparison of responses from the FFRA and NASTRAN modal solution (SOL 111) for FE model 2.

the complex symmetric matrix eigenvalue problem in Step (1) with CSYMM. The CSYMM is almost 12 times faster than the complex general matrix eigensolver, ZGEEV, in LAPACK [11]. Once the complex symmetric matrix eigensolution is obtained, the time for the frequency sweep, 21.7%, is very inexpensive because the coefficient matrix of the modal frequency response problem becomes diagonal. The frequency sweep includes forming $\Phi_C^T \mathbf{F}$ in Step (2.1) and backtransformation for $\mathbf{Z}$ in Step (2.3). The time, which is neglected in the timing profile, is for reading and writing data in disk. This is minimal compared with the other analysis time.

Figures 2a and 2b show the magnitude of the acceleration in the Y-direction at driving point and cross point for the Y-direction excitation force, respectively. The figures and table show the outstanding performance of the FFRA algorithm with good accuracy for a large FE model with structural damping.

V. Conclusion

An efficient algorithm, fast frequency response analysis (FFRA) algorithm, is developed to solve the modal frequency response problem for large structures with structural damping. The FFRA algorithm achieves $\mathcal{O}(m^2)$ operations at each excitation frequency, where m is the number of modes used to represent the response, by reformulating the modal FRA. Therefore, the newly developed algorithm FFRA greatly improves the performance of the damped modal FRA problem compared with existing methods while observing the same accuracy.

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